12 Complementary Subspaces

Complementary Subspaces Subspaces $\mathcal{X}$, $\mathcal{Y}$ of a space $\mathcal{V}$ are said to be complementary whenever

$$\mathcal{V} = \mathcal{X} + \mathcal{Y} \quad \text{and} \quad \mathcal{X} \cap \mathcal{Y} = \mathbf{0},$$

in which case $\mathcal{V}$ is said to be the **direct sum** of $\mathcal{X}$ and $\mathcal{Y}$, and this is denoted by writing $\mathcal{V} = \mathcal{X} \oplus \mathcal{Y}$.

- For a vector space $\mathcal{V}$ with subspaces $\mathcal{X}$, $\mathcal{Y}$ having respective bases $\mathcal{B}_\mathcal{X}$ and $\mathcal{B}_\mathcal{Y}$, the following statements are equivalent.
  - ▷ $\mathcal{V} = \mathcal{X} \oplus \mathcal{Y}$.
  - ▷ For each $v \in \mathcal{V}$ there are unique vectors $x \in X$ and $y \in Y$ such that $\boldsymbol{v} = \boldsymbol{x} + \boldsymbol{y}$.
  - ▷ $\mathcal{B}_\mathcal{X} \cap \mathcal{B}_\mathcal{Y} = \emptyset$ and $\mathcal{B}_\mathcal{X} \cup \mathcal{B}_\mathcal{Y}$ is a basis for $\mathcal{V}$.

1. For a vector space $\mathcal{V}$ with subspaces $\mathcal{X}$, $\mathcal{Y}$ having respective bases $\mathcal{B}_\mathcal{X}$ and $\mathcal{B}_\mathcal{Y}$, prove that the above three statements (in the claim Complementary Subspaces) are equivalent.

2. Let $\mathcal{X}$ and $\mathcal{Y}$ be the subspaces of $\mathbb{R}^3$ that are spanned by

$$\mathcal{B}_\mathcal{X} = \left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \right\} \quad \text{and} \quad \mathcal{B}_\mathcal{Y} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\},$$

respectively. Explain why $\mathcal{X}$ and $\mathcal{Y}$ are complementary, and then determine the projector onto $\mathcal{X}$ along $\mathcal{Y}$. What is the projection of $\boldsymbol{v} = (-2, 1, 3)^\top$ onto $\mathcal{X}$ along $\mathcal{Y}$? What is the projection of $\boldsymbol{v}$ onto $\mathcal{Y}$ along $\mathcal{X}$?

3. Construct an example of a pair of nontrivial complementary subspaces of $\mathbb{R}^5$, and explain why your example is valid.

Projection Suppose that $\mathcal{V} = \mathcal{X} \oplus \mathcal{Y}$ so that for each $\boldsymbol{v} \in \mathcal{V}$ there are unique vectors $\boldsymbol{x} \in \mathcal{X}$ and $\boldsymbol{y} \in \mathcal{Y}$ such that $\boldsymbol{v} = \boldsymbol{x} + \boldsymbol{y}$.

- The vector $\boldsymbol{x}$ is called the projection of $\boldsymbol{v}$ onto $\mathcal{X}$ along $\mathcal{Y}$.
- The vector $\boldsymbol{y}$ is called the projection of $\boldsymbol{v}$ onto $\mathcal{Y}$ along $\mathcal{X}$.

4. Construct an example to show that if $\mathcal{V} = \mathcal{X} + \mathcal{Y}$ but $\mathcal{X} \cap \mathcal{Y} \neq \{\mathbf{0}\}$, then a vector $\boldsymbol{v} \in \mathcal{V}$ can have two different representations as $\boldsymbol{v} = \boldsymbol{x}_1 + \boldsymbol{y}_1$ and $\boldsymbol{v} = \boldsymbol{x}_2 + \boldsymbol{y}_2$, where $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathcal{X}$ and $\boldsymbol{y}_1, \boldsymbol{y}_2 \in \mathcal{Y}$, but $\boldsymbol{x}_1 \neq \boldsymbol{x}_2$ and $\boldsymbol{y}_1 \neq \boldsymbol{y}_2$.

5. Prove that projector P from next statement (claim Projectors) satisfy five given properties.

Projectors Let $\mathcal{X}$ and $\mathcal{Y}$ be complementary subspaces of a vector space $\mathcal{V}$ so that each $\boldsymbol{v} \in \mathcal{V}$ can be uniquely resolved as $\boldsymbol{v} = \boldsymbol{x} + \boldsymbol{y}$, where $x \in \mathcal{X}$ and $\boldsymbol{y} \in \mathcal{Y}$. The unique linear operator P defined by $P\boldsymbol{v} = \boldsymbol{x}$ is called the **projector onto $\mathcal{X}$ along $\mathcal{Y}$**, and P has the following properties.

- $P^2 = P$ (P is idempotent).
- $I - P$ is the complementary projector onto $\mathcal{Y}$ along $\mathcal{X}$.
- $\operatorname{im}(P) = \{\boldsymbol{x} \mid P\boldsymbol{x} = \boldsymbol{x}\}$ (the set of "fixed points" for P).
- $\operatorname{im}(P) = \ker(I - P) = \mathcal{X}$ and $\operatorname{im}(I - P) = \ker(P) = \mathcal{Y}$.
- If $\mathcal{V} = \mathbb{R}^n$ or $\mathbb{C}^n$, then P is given by

  $$P = [X|\mathbf{0}][X|Y]^{-1} = [X|Y] \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} [X|Y]^{-1},$$

  where the columns of $\mathcal{X}$ and $\mathcal{Y}$ are respective bases for $\mathcal{X}$ and $\mathcal{Y}$.

Projectors and Idempotents A linear operator P on $\mathcal{V}$ is a projector if and only if $P^2 = P$.

6. Let $\mathcal{X}$ and $\mathcal{Y}$ be subspaces of R^3 whose respective bases are

$$\mathcal{B}_\mathcal{X} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right\} \quad \text{and} \quad \mathcal{B}_\mathcal{Y} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\},$$

(a) Explain why $\mathcal{X}$ and $\mathcal{Y}$ are complementary subspaces of $\mathbb{R}^3$. (b) Determine the projector P onto $\mathcal{X}$ along $\mathcal{Y}$ as well as the complementary projector Q onto $\mathcal{Y}$ along $\mathcal{X}$. (c) Determine the projection of $\boldsymbol{v} = (2, -1, 1)$ onto $\mathcal{Y}$ along $\mathcal{X}$. (d) Verify that P and Q are both idempotent. (e) Verify that $\operatorname{im}(P) = \mathcal{X} = \ker(Q)$ and $\ker(P) = \mathcal{Y} = \operatorname{im}(Q)$.

7. Explain why $\operatorname{Mat}_{n\times n}(\mathbb{R}) = \mathcal{S} \oplus \mathcal{K}$, where $\mathcal{S}$ and $\mathcal{K}$ are the subspaces of $n \times n$ symmetric and skew-symmetric matrices, respectively. What is the projection of $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, onto $\mathcal{S}$ along $\mathcal{K}$?

8. For a general vector space, let $\mathcal{X}$ and $\mathcal{Y}$ be two subspaces with respective bases $\mathcal{B}_\mathcal{X} = \{\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_m\}$ and $\mathcal{B}_\mathcal{Y} = \{\boldsymbol{y}_1, \boldsymbol{y}_2, ..., \boldsymbol{y}_n\}$. (a) Prove that $\mathcal{X} \cap \mathcal{Y} = \mathbf{0}$ if and only if $\{\boldsymbol{x}_1, ..., \boldsymbol{x}_m, \boldsymbol{y}_1, ..., \boldsymbol{y}_n\}$ is a linearly independent set. (b) Does $\mathcal{B}_\mathcal{X} \cup \mathcal{B}_\mathcal{Y}$ being linear independent imply

$\mathcal{X} \cap \mathcal{Y} = \mathbf{0}$? (c) If $\mathcal{B}_{\mathcal{X}} \cup \mathcal{B}_{\mathcal{Y}}$ is a linearly independent set, does it follow that $\mathcal{X}$ and $\mathcal{Y}$ are complementary subspaces? Why?

9. Let P be a projector defined on a vector space $\mathcal{V}$. Prove that the range of a projector is the set of its "fixed points" in the sense that $\mathrm{im}(P) = \{\boldsymbol{x} \in \mathcal{V} \mid P\boldsymbol{x} = \boldsymbol{x}\}$.

10. Suppose that $\mathcal{V} = \mathcal{X} \oplus \mathcal{Y}$, and let P be the projector onto $\mathcal{X}$ along $\mathcal{Y}$. Prove that

$$\mathrm{im}(P) = \ker(I - P) = \mathcal{X} \qquad \text{and}$$

$$\mathrm{im}(I - P) = \ker(P) = \mathcal{Y}.$$

11. Suppose that $\mathbb{R}^n = \mathcal{X} \oplus \mathcal{Y}$, where $\dim \mathcal{X} = r$, and let P be the projector onto $\mathcal{X}$ along $\mathcal{Y}$. Explain why there exist matrices $X \in \mathrm{Mat}_{n\times r}(\mathbb{R})$ and $A \in \mathrm{Mat}_{r\times n}(\mathbb{R})$ such that $P = XA$, where $\mathrm{rank}(X) = \mathrm{rank}(A) = r$ and $AX = I$. This is a *full-rank factorization* for P.

12. For either a real or complex vector space, let E be the projector onto $\mathcal{X}_1$ along $\mathcal{Y}_1$, and let F be the projector onto $\mathcal{X}_2$ along $\mathcal{Y}_2$. Prove that $E + F$ is a projector if and only if $EF = FE = 0$, and under this condition, prove that $\mathrm{im}(E + F) = \mathcal{X}_1 \oplus \mathcal{X}_2$ and $\ker(E + F) = \mathcal{Y}_1 \cap \mathcal{Y}_2$.

13. Let $\mathcal{M}$ be subspace of $\mathbb{R}^4$ defined with

$$\mathcal{M} = \{(z_1, z_2, z_3, z_4)^\top \in \mathbb{R}^4 \mid z_1 + 2z_2 + z_3 = 0,$$

$$2z_1 + z_2 - z_3 = 0, z_1 + 5z_2 + 4z_3 = 0\}.$$

Find $\mathcal{N}$ such that $\mathcal{M}$ and $\mathcal{N}$ are complementary subspaces of a space $\mathbb{R}^4$.

14. In vector space $\mathbb{R}^5$, let $\mathcal{M}$ be subspace spanned by $(0, 0, 1, 0, 0)^\top$ and $(0, 1, 0, 1, 0)^\top$ and let

$$\mathcal{L} = \{(x_1, x_2, x_3, x_4, x_5)^\top \in \mathbb{R}^5 \mid x_1 - x_2 + x_3 = 0,$$

$$2x_1 - 2x_2 + x_3 + x_4 = 0\}.$$

(a) Find a basis and dimensions for $\mathcal{M}$ and $\mathcal{L}$. (b) Find a dimension of subspace $\mathcal{M} \cap \mathcal{L}$. (c) Find a basis for complementary subspace $\mathcal{K}$ of the space $\mathcal{L}$ (i.e. find a basis for subspace $\mathcal{K}$ where $\mathcal{L}$ and $\mathcal{K}$ are complementary subspaces of a space $\mathbb{R}^5$).

15. Let $Q : \mathcal{P}_3 \to \mathcal{P}_3$ ($\mathcal{P}_3$ is a vector space of all polynomials of degree ≤ 3) denote a given linear operator defined with

$$Q(p) = \text{all polynomials of degree 2 which graph}$$

$$\text{pass through the points } (-1; p(-1)),\ (0; p(0))$$

$$\text{and } (1; p(1)).$$

(a) Find coordinate matrix of Q with the respect to the standard basis. (b) Find complementary subspace $\mathcal{N}$ of the space $\mathcal{M} = \ker(Q)$ in $\mathcal{P}_3$.

16. Let

$$\mathcal{L} = \{(x_1, x_2, x_3, x_4)^\top \in \mathbb{R}^4 \mid -x_1 + x_2 + x_3 + x_4 = 0,$$

$$x_1 - x_2 + x_3 + x_4 = 0,\ x_1 + x_2 - x_3 + x_4 = 0,$$

$$x_1 + x_2 + x_3 - x_4 = 0\}$$

denote a given set. Show that $\mathcal{L}$ is a subspace of $\mathbb{R}^4$, find a basis, dimension and find complementary subspace of $\mathcal{L}$ in $\mathbb{R}^4$.

17. In a vector space $\mathcal{P}_4$ of all real polynomials of degree ≤ 4 it is given a set

$$\mathcal{M} = \{p \in \mathcal{P}_4 \mid p'(0) = p(1),\ p''(0) = 2p(-1)\}.$$

Show that $\mathcal{M}$ is a vector subspace of $\mathcal{P}_4$, find a basis and dimension, and find complementary subspace of $\mathcal{M}$ in $\mathcal{P}_4$.

Historical remark: Jacobi[10] Reduction. Can we construct an orthogonal matrix P such that $P^\top AP = D$ is a diagonal matrix? Indeed we can, and much of the material concerning eigenvalues and eigenvectors is devoted to this problem. This fact can be constructively established by means of Jacobi's diagonalization algorithm (see page 353 of the book).

[10]Karl Gustav Jacob Jacobi (1804–1851) first presented this method in 1846, and it was popular for a time. But the twentieth-century development of electronic computers sparked tremendous interest in numerical algorithms for diagonalizing symmetric matrices, and Jacobi's method quickly fell out of favor because it could not compete with newer procedures-at least on the traditional sequential machines. However, the emergence of multiprocessor parallel computers has resurrected interest in Jacobi's method because of the inherent parallelism in the algorithm. Jacobi was born in Potsdam, Germany, educated at the University of Berlin, and employed as a professor at the University of Königsberg. During his prolific career he made contributions that are still important facets of contemporary mathematics. His accomplishments include the development of elliptic functions; a systematic development and presentation of the theory of determinants; contributions to the theory of rotating liquids; and theorems in the areas of differential equations, calculus of variations, and number theory. In contrast to his great contemporary Gauss, who disliked teaching and was anything but inspiring, Jacobi was regarded as a great teacher (the introduction of the student seminar method is credited to him), and he advocated the view that "the sole end of science is the honor of the human mind, and that under this title a question about numbers is worth as much as a question about the system of the world." Jacobi once defended his excessive devotion to work by saying that "Only cabbages have no nerves, no worries. And what do they get out of their perfect wellbeing?" Jacobi suffered a breakdown from overwork in 1843, and he died at the relatively young age of 46.